

$$1) \mathcal{L}(x) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\Psi} (i \gamma^\mu D_\mu - m_e) \Psi + (D^\nu \phi)^* (D_\nu \phi) - m_\phi^2 \phi^* \phi$$

Since $D_\mu = \partial_\mu + ieA_\mu$ contains the EM field A_μ , it is responsible for introducing interactions among the fields. This Lagrangian density therefore describes interactions between fermions and photons, as well as between photons and some scalar particle ϕ .

$$\text{Let } \mathcal{L}_{\text{fermi}}(x) \equiv \bar{\Psi} (i \gamma^\mu D_\mu - m_e) \Psi.$$

$$\begin{aligned} \Rightarrow \mathcal{L}_{\text{fermi}}(x) &= \bar{\Psi} i \gamma^\mu D_\mu \Psi - m_e \bar{\Psi} \Psi \\ &= i \bar{\Psi} \gamma^\mu (\partial_\mu + ieA_\mu) \Psi - m_e \bar{\Psi} \Psi \\ &= i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - e \bar{\Psi} \gamma^\mu A_\mu \Psi - m_e \bar{\Psi} \Psi \\ &= \bar{\Psi} (i \gamma^\mu \partial_\mu - m_e) \Psi - e \bar{\Psi} \gamma^\mu \Psi A_\mu \end{aligned}$$

$$\text{Now let } \mathcal{L}_{\text{scalar}}(x) \equiv (D^\nu \phi)^* (D_\nu \phi)$$

$$\begin{aligned} \Rightarrow \mathcal{L}_{\text{scalar}}(x) &= ((\partial^\nu + ieA^\nu) \phi)^* ((\partial_\nu + ieA_\nu) \phi) \\ &= (\partial^\nu - ieA^\nu)^* \phi^* (\partial_\nu + ieA_\nu) \phi \end{aligned}$$

$$= (\partial^\mu \phi^* - ie A^\mu \phi^*) (\partial_\mu \phi + ie A_\mu \phi)$$

$$= (\partial^\mu \phi^*) (\partial_\mu \phi) + ie (\partial^\mu \phi)^* A_\mu \phi - ie A^\mu \phi^* (\partial_\mu \phi) + e^2 A^\mu \phi^* A_\mu \phi$$

Now I'll pick the interaction terms out of $\mathcal{L}_{\text{fermi}}$ and $\mathcal{L}_{\text{scalar}}$ and label them \mathcal{L}_I .

$$\mathcal{L}_I(x) = -e \bar{\psi} \gamma^\mu \psi A_\mu + ie ((\partial^\mu \phi)^* A_\mu \phi - (A^\mu \phi)^* \partial_\mu \phi) + e^2 |A|^2 |\phi|^2$$

The general definition for the interaction Hamiltonian is

$$\mathcal{H}_I(x) = \pi(x) \dot{\phi}(x) - \mathcal{L}_I(x)$$

where $\pi(x)$ is a momentum field conjugate to $\phi(x)$.

Notice that there are some $\dot{\phi}(x)$ terms in the 2nd term of $\mathcal{L}_I(x)$, but Prof. Han says these can be ignored, so

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x)$$

$$\mathcal{H}_I(x) = e \bar{\psi} \gamma^\mu \psi A_\mu - ie ((\partial^\mu \phi)^* A_\mu \phi - (A^\mu \phi)^* \partial_\mu \phi) - e^2 |A|^2 |\phi|^2$$

$$2) e^-(p_-) + e^+(p_+) \rightarrow h(q_-) + H^+(q_+)$$

The interaction part of the scattering matrix is

$$\langle q_-; q_+ | i T | p_-; p_+ \rangle = \lim_{t \rightarrow \alpha(1-i\epsilon)} \langle q_-; q_+ | T \exp \left[-i \int_{-t}^t dt' H_I(t') \right] | p_-; p_+ \rangle$$

↑
time-ordering

where the exponential expands as

$$T \exp \left[-i \int_{-t}^t dt' H_I(t') \right] = 1 + (-i) \int_{-t}^t dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{-t}^t dt_1 dt_2 T[H_I(t_1) H_I(t_2)] + \dots$$

As we are beginning and ending with two particle states, the 2nd order term of the expansion is the first to contribute to the specified interaction.

The scattering matrix now becomes

$$\begin{aligned} & \approx \left\{ \langle q_-; q_+ | \frac{(-i)^2}{2!} \int_{-t}^t dt_1 dt_2 T[H_I(t_1) H_I(t_2)] | p_-; p_+ \rangle \right. \\ & \quad \left. H_I(t) = \int d^3x \left[e \bar{\psi} \gamma^\mu \psi A_\mu - ie \underbrace{((\partial^\mu \phi)^* A_\mu \phi - (A^\mu \phi)^* \partial_\mu \phi)}_{= \mathcal{H}_\phi} - e^2 |A|^2 |\phi|^2 \right] \right\} \\ & \quad \equiv \mathcal{H}_{QED} \quad \equiv \mathcal{H}_\phi \quad \equiv \mathcal{H}_{\phi^2} \end{aligned}$$

$$= \langle q_-; q_+ | \frac{(-i)^2}{2!} \int_{-t}^t dt_1 dt_2 T \left[\int d^3x_1 \mathcal{H}(x_1) \int d^3x_2 \mathcal{H}(x_2) \right] | p_-; p_+ \rangle$$

$$= \langle q_f^-; q_f^+ | \frac{(-i)^2}{2!} T \left[\int d^4x_1 \delta(x_1) \left\{ \int d^4x_2 \delta(x_2) \right\} \right] | p_-; p_+ \rangle$$

$$= \frac{(-i)^2}{2!} \left\{ \int d^4x_1 d^4x_2 \langle q_f^-; q_f^+ | T \left[\delta(x_1) \delta(x_2) \right] | p_-; p_+ \rangle \right.$$

$$= \frac{(-i)^2}{2!} \left\{ \int d^4x_1 d^4x_2 \langle q_f^-; q_f^+ | T \left[\cancel{\delta_{QED}(x_1)} \cancel{\delta(x_2)} + \cancel{\delta(x_1)} \cancel{\delta_{QED}(x_2)} + \cancel{\delta(x_1)} \cancel{\delta(x_2)} \right] | p_-; p_+ \rangle \right.$$

$$\left. + \cancel{\delta(x_1)} \cancel{\delta(x_2)} + \cancel{\delta(x_1)} \cancel{\delta(x_2)} + \cancel{\delta(x_1)} \cancel{\delta(x_2)} \right. \\ \left. + \cancel{\delta(x_1)} \cancel{\delta(x_2)} + \cancel{\delta(x_1)} \cancel{\delta(x_2)} + \cancel{\delta(x_1)} \cancel{\delta(x_2)} \right] | p_-; p_+ \rangle$$

Keeping in mind the physical process, we can rule out contributions from most of these terms. I need $\delta(x_1) \delta(x_2)$ terms with six fields: two to contract with the initial states, two to contract with the final states, and two to contract with each other to form a propagator from x_1 to x_2 . All such terms with an $\cancel{\delta}$ are ruled out because they have seven terms. The $\cancel{\delta_{QED}(x_1)} \cancel{\delta_{QED}(x_2)}$ term is ruled out because it does not contain any scalar fields. The $\cancel{\delta(x_1)} \cancel{\delta(x_2)}$ term is ruled out because it does not contain any fermionic fields.

All that's left is

$$= \frac{(-i)^2}{2!} \left\{ \int d^4x_1 d^4x_2 \langle q_- q_+ | T \left[\underbrace{\partial_\mu \phi_1(x_1) \partial_\nu \phi_1(x_2)}_{\text{QED}} + \underbrace{\partial_\mu \phi_1(x_1) \partial_\nu \phi_2(x_2)}_{\text{QED}} \right] \right|_{(P_-; P_+)}$$

$\equiv \text{Term I}$ $\equiv \text{Term II}$

Before proceeding, I need to make a digression into how derivatives of scalar fields ϕ contract with momentum states.

Recall that

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a(\vec{p}) e^{-ip \cdot x} + a^\dagger(\vec{p}) e^{ip \cdot x} \right]$$

and $\overline{\phi(x)} |\vec{p}'\rangle = e^{-ip \cdot x} |0\rangle$

from eqn. (4.93).

$$\begin{aligned} \Rightarrow \partial^\mu \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \left[a(\vec{p}) \partial^\mu e^{-ip \cdot x} + a^\dagger(\vec{p}) \partial^\mu e^{ip \cdot x} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \left[a(\vec{p})(-ip^\mu) e^{-ip \cdot x} + a^\dagger(\vec{p})(ip^\mu) e^{ip \cdot x} \right] \\ &= (-ip^\mu) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \left[a(\vec{p}) e^{-ip \cdot x} - a^\dagger(\vec{p}) e^{ip \cdot x} \right] \end{aligned}$$

How does this contract w/ a momentum-state?

$$\begin{aligned}
\langle \bar{\psi} \phi | \bar{p}' \rangle &= (-i p'^\mu) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a(\vec{p}) e^{-ip \cdot \vec{x}} - a^\dagger(\vec{p}) e^{ip \cdot \vec{x}} \right] \sqrt{2E_{p'}} a_{p'}^\dagger | 0 \rangle \\
&= (-i p'^\mu) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a(\vec{p}) e^{-ip \cdot \vec{x}} \sqrt{2E_{p'}} a_{p'}^\dagger | 0 \rangle \\
&= (-i p'^\mu) \int \frac{d^3 p}{(2\pi)^3} \frac{\sqrt{2E_p}}{\sqrt{2E_{p'}}} e^{-ip \cdot \vec{x}} (2\pi)^3 \delta(\vec{p} - \vec{p}') | 0 \rangle \\
\underline{\bar{\psi} \phi | \bar{p}' \rangle = (-i p'^\mu) e^{-ip \cdot \vec{x}} | 0 \rangle}
\end{aligned}$$

Back to the interacting part of the scattering matrix.

Term I

$$\begin{aligned}
&\frac{(-i)^2}{2!} \iint d^4x_1 d^4x_2 \langle q_-; q_+ | T \left[\frac{\gamma^\mu(x_1) \gamma^\nu(x_2)}{q_1} \right] | p_-; p_+ \rangle \\
&= \frac{(-i)^2}{2} \iint d^4x_1 d^4x_2 \langle q_-; q_+ | T \left[-ie^2 \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1) (\partial_\nu^\mu \phi)^* A_\nu(x_2) \phi(x_2) - A_\nu(x_2) \phi^*(x_2) \partial_\nu^\mu \phi(x_2) \right] | p_-; p_+ \rangle \\
&= \frac{(-i)^2}{2} \iint d^4x_1 d^4x_2 \langle q_-; q_+ | T \left[-ie^2 \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1) (\partial_\nu^\mu \phi(x_2))^* A_\nu(x_2) \phi(x_2) \right] | p_-; p_+ \rangle \\
&+ \frac{(-i)^2}{2} \iint d^4x_1 d^4x_2 \langle q_-; q_+ | T \left[ie^2 \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1) A_\nu^*(x_2) \phi^*(x_2) \partial_\nu^\mu \phi(x_2) \right] | p_-; p_+ \rangle
\end{aligned}$$

Now I will calculate both of these time-ordered products.

All terms with fields of different types contracted vanish. If a field contracts with

another of the same kind of field at the same spacetime coordinate, then it forms a loop and there is no propagation from x_1 to x_2 (or vice versa). Terms with more than one contraction do not contribute to this interaction because there would not be enough fields left over to contract with the initial and final states. Thus, the only term left is the one that propagates a photon from x_1 to x_2 (or vice versa).

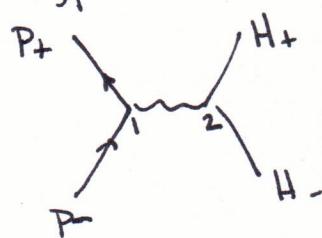
$$T[\bar{\psi}_1 \gamma^\mu \psi_1 A_{1\mu} (\partial^2 \phi_2)^* A_{2\mu} \phi_2] = N \{ \bar{\psi}_1 \gamma^\mu \psi_1 A_{1\mu} \overline{(\partial^2 \phi_2)^* A_{2\mu} \phi_2} \}$$

Similarly,

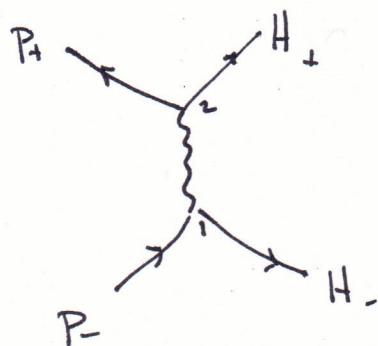
$$T[\bar{\psi}_1 \gamma^\mu \psi_1 A_{1\mu} A_{2\mu}^{**} \phi_2^* (\partial_\mu \phi_2)] = N \{ \bar{\psi}_1 \gamma^\mu \psi_1 A_{1\mu} \overline{A_{2\mu}^{**} \phi_2^* (\partial_\mu \phi_2)} \}$$

At this point, something can already be said of the diagrams. Notice how spacetime position x_1 has fermion fields, but no scalar fields. Likewise, spacetime position x_2 has scalar fields, but no fermion fields. This means

we are presently looking at an interaction
of the type



but not of the type

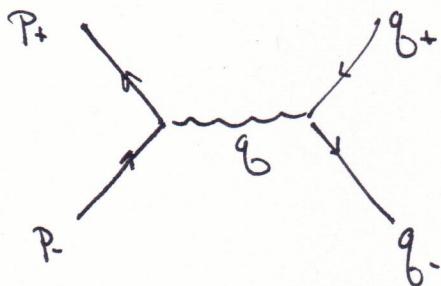


Let's get back to using the time-ordered
products in the scattering matrix.

$$\begin{aligned}
 & \frac{(-e)^2}{2} \iint d^4x_1 d^4x_2 \langle q^-; q_+ | \cancel{q}_- T [\cancel{q}_1^\mu(x_1) \cancel{q}_2^\nu(x_2)] | P^-; P_+ \rangle \\
 &= \frac{ie^2}{2} \iint d^4x_1 d^4x_2 \langle q^-; q_+ | \overbrace{\cancel{q}_-^\mu(x_1) \gamma^\nu \cancel{q}_-^\nu(x_2)}^{\text{QED}} \overbrace{A_\mu(x_1) (\partial_\nu \phi(x_2))^*}^{(1)} \overbrace{A_\nu(x_2) \phi(x_1)}^{(2)} | P^-; P_+ \rangle \\
 & - \frac{ie^2}{2} \iint d^4x_1 d^4x_2 \langle q^-; q_+ | \overbrace{\cancel{q}_-^\mu(x_1) \gamma^\nu \cancel{q}_-^\nu(x_2)}^{\text{QED}} \overbrace{A_\mu^*(x_1) \partial_\nu^* \phi^*(x_2) (\partial_\nu \phi(x_1))}^{(1)} | P^-; P_+ \rangle
 \end{aligned}$$

$$\begin{aligned}
&= \frac{i e^2}{2} \iint d^4x_1 d^4x_2 e^{i p_+ \cdot x_1} \bar{v}^a(p_+) \gamma^\mu e^{i p_- \cdot x_2} u^a(p_-) \int \frac{dq}{(2\pi)^4} \left[\frac{-ig_{\mu\nu}}{q_f^2 + i\varepsilon} \right] e^{-iq \cdot (x_1 - x_2)} (-iq^{\nu}) e^{-iq_+ \cdot x_1} e^{-iq_- \cdot x_2} \\
&\quad - \frac{i e^2}{2} \iint d^4x_1 d^4x_2 e^{i p_+ \cdot x_1} \bar{v}^a(p_+) \gamma^\mu e^{i p_- \cdot x_2} u^a(p_-) \int \frac{dq}{(2\pi)^4} \left[\frac{-ig_{\mu\nu}}{q_f^2 + i\varepsilon} \right] e^{-iq \cdot (x_1 - x_2)} (iq^{\nu}) e^{-iq_+ \cdot x_1} e^{-iq_- \cdot x_2} \\
&= -\frac{i e^2}{2} \iint d^4x_1 d^4x_2 \bar{v}^a(p_+) \gamma^\mu u^a(p_-) e^{i(p_+ + p_- - q) \cdot x_1} \int \frac{dq}{(2\pi)^4} \left[\frac{g_{\mu\nu}}{q_f^2 + i\varepsilon} \right] q^{\nu} e^{i(q - q_+ - q_-) \cdot x_2} \\
&\quad - \frac{i e^2}{2} \iint d^4x_1 d^4x_2 \bar{v}^a(p_+) \gamma^\mu u^a(p_-) e^{i(p_+ + p_- - q) \cdot x_1} \int \frac{dq}{(2\pi)^4} \left[\frac{g_{\mu\nu}}{q_f^2 + i\varepsilon} \right] q^{\nu} e^{i(q - q_+ - q_-) \cdot x_2} \\
&= -i e^2 \int dq_f \bar{v}^a(p_+) \gamma^\mu u^a(p_-) \left(\frac{g_{\mu\nu}}{q_f^2 + i\varepsilon} \right) q^{\nu} \delta(p_+ + p_- - q_f) \delta(q_f - q_+ - q_-) \\
&= \boxed{-i e^2 \bar{v}^a(p_+) \gamma^\mu u^a(p_-) \left(\frac{g_{\mu\nu}}{q_f^2 + i\varepsilon} \right) q^{\nu}} \text{ for } q = p_+ + p_- = q_+ + q_-
\end{aligned}$$

3) As shown on the previous page, the Feynman Diagram is



The Feynman Rules are a mix of the usual fermionic and scalar rules.

a) The momentum-space EM field propagator is

$$\frac{-ig}{q^2 + i\epsilon}$$

b) The photon-fermion vertex factor is

$$-ie\gamma^\mu$$

c) The photon-scalar vertex factor is

$$e$$

d) Fermionic external contractions:

$$\overline{\psi} |p\rangle = u(p)$$

$$\bar{\psi} |p\rangle = \bar{v}(p)$$

e) Scalar external contractions

$$\langle q | d = 1$$

$$\langle q | (\partial^\mu \phi) = q^\mu \quad (\text{solved for earlier p. 6})$$

f) Impose conservation of 4-momenta at vertices w/ δ -function

g) Integrate over the propagator momenta. In this case, q .

These steps will reproduce the amplitude.